Part TWO

Mathematical Methods for Physics

from **On the Studies of Physics and Her Axillary Studies** by Shing Hin (John) Yeung

Chapter 27

Algebra on the Complex Set

What is in this chapter?

Complex numbers are very important in describing, for example, wave functions and potentials. In this chapter, we will be exploring the basic arithmetic of complex numbers.

Chapter 27 Algebra on the Complex Set

27.1 Complex (Number) Set is the Largest Numerical Set

Textbooks when introducing complex number, do list a set of equations and specify the largest set could not solve such equation. Hence we ought for a complex number. In Physics, since it is known that complex numbers are able to solve all of the polynomial equations, the properties upon each numerical set determines the numerical set for choice.

Figure 27.1 showing the relationships among all numerical sets, yet specify why each set is a subset to the other. Such argument comes from the larger set has the same function as its subsets, while they are unique on where excluded from their subset(s). For example,

1. **Natural numbers** ($\mathbb{N}$) are composed by positive integers, sometimes exclude zero.
2. **Integers** ($\mathbb{Z}$) include negative integers, apart from all numbers in the set of $\mathbb{N}$.
3. **Rational** ($\mathbb{Q}$) can be presented in terms of fractions.
4. **Real numbers** ($\mathbb{R}$) includes infinite decimals, hence cannot be represented by fractions.
5. **Complex numbers** ($\mathbb{C}$) includes a term of i, the imaginary unit.

27.1.1 The Magic Number: $\sqrt{-1} = i$

The **definition of an imaginary unit** is

$$i^2 = -1 \tag{27.1}$$

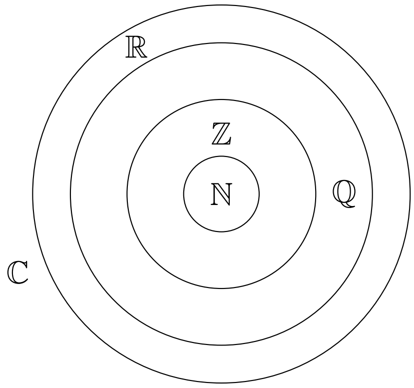

Figure 27.1: Venn diagram of all number systems. While all number sets are presented in rings, they include their proper subsets (e.g. real numbers $\mathbb{R}$ are proper subsets of complex numbers $\mathbb{C}$). Note that this is the replication of fig. 26.1.

and any powers of the imaginary unit will become as either one or an imagine unit, and possibly with different parities. For example,

$$\begin{aligned} i^0 &= 1 \\ i^1 &= i \\ i^2 &= -1 \\ i^3 &= -i \\ i^4 &= 1 \\ \vdots &= \vdots \\ i^n &= i^n \pmod 4 \end{aligned} \tag{27.2}$$

27.1.2 Complex Numbers are Composed with two Components

A complex number is composed of an imaginary part and/ or a real part. With the form of

$$z = \mathrm{Re}(z) + \mathrm{Im}(z)^{1} : z \in \mathbb{C} \tag{27.3}$$

The difference between **the real component** $\mathrm{Re}(z)$ and **the imaginary component** $\mathrm{Im}(z)$ is the **latter contains an imaginary unit** i. However, we omit the imaginary unit due to conventions.

The real part comes from the real number set. Hence z without the imaginary part, and not specified the system which may accepts complex field, does not explicitly counted as a complex number. The imaginary part consists of a product with the imaginary unit eq. (27.1).

[1] $\mathrm{Im}(z)$ is defined with an imaginary unit, no matter a positive or negative (i.e. $-i$).

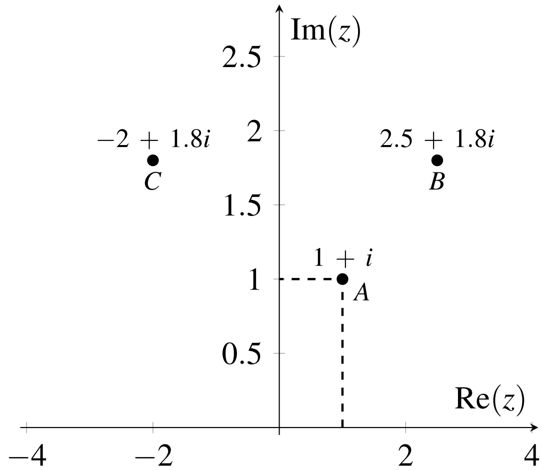

Figure 27.2: Argand diagram representing $z = \text{Re}(z) + \text{Im}(z)$

Exercise 27.1.1: Components of Complex Numbers

Let $z = x + iy$, find $\text{Re}(z)$ and $\text{Im}(z)$.

Solution:

The real component of z is

$$\text{Re}(z) = x \tag{27.4}$$

The imaginary component of z is

$$\text{Im}(z) = y \tag{27.5}$$

where we omit the imaginary unit, the compulsory component of $\text{Im}(z)$.

27.1.3 Properties of Complex Numbers

Complex conjugates becomes handful when referencing with its conjugate. In terms of the properties of the (complex conjugate) operator

Complex Numbers are Commutative The sums of complex conjugates does not depend on their order, that is

$$z_1 + z_2 = z_2 + z_1 \tag{27.6}$$

The products of complex conjugates does not depend on their order, that is

$$z_1 z_2 = z_2 z_1 \tag{27.7}$$

Where in eqs. (27.6) and (27.7) does not limit on the numbers of terms.

Complex Numbers are Associative The sums of complex numbers does not depend on their order of computation, that is

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \tag{27.8}$$

The products of complex numbers does not depend on their order of computation, that is

$$(z_1 z_2) z_3 = z_1 (z_2 z_3) \tag{27.9}$$

Complex Numbers are Distributive Complex numbers are both left and right distributive, such as

$$w(z_1 + z_2) = (w z_1) + (w z_2) \quad \text{(Left distributive)} \tag{27.10}$$

$$(z_1 + z_2) w = (z_1 w) + (z_2 w) \quad \text{(Right distributive)} \tag{27.11}$$

where eq. (27.7) implies that eqs. (27.10) and (27.11) are equivalent.

Negative Complex Number When the two components carries the opposite parity, but same magnitude[2]. Both complex numbers sits in opposite (in Argand Diagram).

Zero Complex Number There exit a complex number that act as the same as integer zero (0). This is explained in theorem 27.1.1.

27.1.1 Theorem (Zero Complex Number)

When the real and imaginary components are zero. Then $z = 0 \in \mathbb{C}$.

For example, the origin point O is the zero complex number in an Argand Diagram.

27.1.1.1 Corollary (Adding Zeros to Complex Numbers)

Let any complex number z, then any $z + 0 \equiv z$.

27.1.1.2 Corollary (Summation of Opposite Complex Numbers returns to Zero)

Two complex numbers, one contains opposite parity of real and imaginary components. When the two adds together, such summation returns zero.

$$z + (-z) = 0 \tag{27.12}$$

[2] Not to confuse with complex magnitude: This is the magnitude of each components.

The Complex Number of One The integer one (1)[3][4] when multiply with any complex numbers, will return to all products without the number one (1). This is same as other numeral sets. Since the number of one (1) may decomposed as its equivalent fraction

$$1 = \underbrace{\frac{1}{1} = \frac{w}{w}}_{\text{Equivalent}} : w \in \mathbb{C} \tag{27.13}$$

eq. (27.13) is useful in rationalising fractions involving complex numbers in the denominator.

Ratios of the Magnitudes of Complex Numbers When two complex numbers are defined, and we want to compute the ratio of such. The following theorem simplifies when using the Polar coordinates.

27.1.2 Theorem (Expressing the Ratios of the Complex Magnitudes)
The magnitude of two complex numbers in ratio, is the ratio of two complex magnitudes. That is

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \tag{27.14}$$

This is useful when computing in Polar coordinates.

27.1.4 The Conjugate of a Complex Number contains Opposite Parity to its Imaginary Component

The conjugate of a complex number z is denoted as $z*$, and defined as

$$z* \ : \ \mathrm{Im}(z*) = -\mathrm{Im}(z) \tag{27.15}$$

27.1.1 Definition (Complex Conjugate)
The conjugate of a complex number z is denoted as $z*$, where the conjugates share the imaginary component with opposite parities.

while it is more popular to denote

$$\bar{z} \quad \text{(The more popular version to denote complex conjugate.)}^{5} \tag{27.16}$$

this book will use eq. (27.15) to avoid confusion with the mean of samples.

Complex conjugates are very useful. As eq. (27.15) implies, addition of two (2) complex numbers will omit their corresponding imaginary components.

[3] Figure 27.1 shows that number set $\mathbb{C}$ contains $\mathbb{Z}$.
[4] Note that one (1) have an imaginary component as zero.
[5] The author uses this while handwriting, since it is unlikely to have complex numbers and statistics in the same regime.

Complex conjugates are also useful in rationalise fractions with complex numbers in its denominators. Where this is done by the product of the conjugate pairs $z \cdot z*$ where the imaginary part (in z) remains but the imaginary unit cancels the mixed term[6] and changes the parity of the imaginary component. We will be looking into such further here.

Exercise 27.1.2: Complex Conjugate of Conventional Complex Number

Use z where from Exercise 27.1.1, hence find $z*$.

Solution:
The real component of z is x, from Exercise 27.1.1 states.
The imaginary component of z is y, from Exercise 27.1.1 states. The connector between the components is an addition, hence by eq. (27.15)

$$z* = x - iy \tag{27.17}$$

Exercise 27.1.3: Complex Conjugate of Conjugated Complex Number

Use the outcome from Exercise 27.1.2, hence find the complex conjugate $z*$.

Solution:
First of all, $z*$ is an operator for changing the parity of the imaginary component. Hence to avoid confusion, let $w \in \mathbb{C}$ as the outcome of eq. (27.17).
The real component of w is x, from eq. (27.17) states. The imaginary component of w is $-y$, from eq. (27.17) states. The connector between the components is an subtraction, hence by eq. (27.15)

$$w* = x + iy = z \tag{27.18}$$

where z corresponds to Exercise 27.1.1.

As such expression is done after two of the same operators, we may say the complex conjugates are invertible. That is

$$(z*)* = z \tag{27.19}$$

27.1.5 Real and Imaginary Components can be Composed by Complex Conjugate Pairs

Given from definition 27.1.1 that conjugate pairs differ in the parities of their imaginary components. By combining both will obtain the real or imaginary

[6] Note that $z \cdot z*$ produces a squared term.

component.

Note that conjugate pairs **must have the same magnitude for both real and imaginary components**.

The Summation of Conjugate Pairs is Doubled the Real Component Since conjugate pairs are differ in the parities of the imaginary component, but adding the pair will cancel the imaginary component. Hence,

$$\mathrm{Re}(z) = \frac{z + z*}{2} \tag{27.20}$$

The Subtraction of Conjugate Pairs is Doubled the Imaginary Component Where for the imaginary component, we require the magnitude of such. Hence,

$$\mathrm{Im}(z) = \frac{z - z*}{2i} \tag{27.21}$$

is useful. Given that imaginary components must companion with an imaginary unit i.

27.1.6 Properties of Complex Conjugates

Complex conjugates becomes handful when referencing with its conjugate. In terms of the properties of the (complex conjugate) operator

Complex Conjugates are Commutative The sums of complex conjugates does not depend on their order, that is

$$(z_1 + z_2)* = (z_2 + z_1)* \tag{27.22}$$

The products of complex conjugates does not depend on their order, that is

$$(z_1 z_2)* = (z_2 z_1)* \tag{27.23}$$

Where in eqs. (27.22) and (27.23) does not limit on the numbers of terms.

Complex Conjugate Operators are Associative on Algebraic Operations Think complex conjugate as a operator. Then such operator, when acting upon the summation or products of complex numbers, does not affect the order to compute the operator or the algebra. Hence,

27.1.3 Theorem (Complex Conjugate Operators: Associativity with Summation) The operation of either apply the complex conjugate operator or summations of all complex numbers are invariant. That is,

$$\left(\sum_k z_k\right)* = \sum_k z*_k \tag{27.24}$$

27.1.4 Theorem (Complex Conjugate Operators: Associativity with Product)

The operation of either apply the complex conjugate operator or multiplying all complex numbers are invariant. That is,

$$\left(\prod_k z_k\right)* = \prod_k z*_k \tag{27.25}$$

Theorems 27.1.3 and 27.1.4 means we should not be worried which method to use. Sometimes when one method is preferred, it could be the left hand side of eqs. (27.24) and (27.25) refrains to apply the operator onto each complex number, or the right hand side is good for when each complex number is the target.

This is useful in computational methods.

Complex Conjugate has the Magnitude of its Conjugate The heading says conjugate pairs has the same magnitude, given that opposing the imaginary component solely changes the argument. Furthermore, a pictorial description is shown in fig. 27.3.

Product of Conjugate Pairs is the Squared Complex Magnitude The product of conjugate pairs $z \cdot z*$ is the squared complex magnitude. That is,

$$z \cdot z* = |z|^2 \tag{27.26}$$

where $z = \mathrm{Re}(z) + \mathrm{Im}(z)$. The return is a complex number with zero imaginary component. Hence $|z|^2 = |x|^2 + |y|^2$ if $z = x + iy$.

On the Argand Diagram, $z \cdot z*$ is shown as a vector projected on the real component. A pictorial description is shown in fig. 27.3.

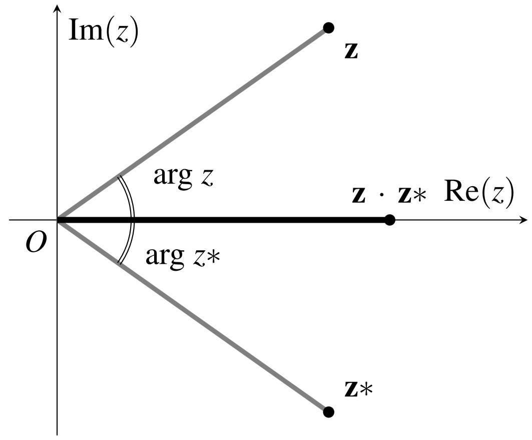

Figure 27.3: $z \cdot z*$ on Argarnd Diagram. Note that $|z|$ and $|z*|$ are equivalent, while $-\arg(z*) = \arg(z)$.

27.2 Addition and Subtraction of Complex Numbers is done by Per-component Basis

Adding complex numbers is done by **the sum of all real components plus the sum of all imaginary components**. That is,

$$\sum_k z_k = \sum_k \mathrm{Re}\,(z_k) + \mathrm{Im}\,(z_k)^{7} \tag{27.27}$$

As a reminder, note that in eqs. (27.3) and (27.27) and fig. 27.2, the imaginary unit is part of the imaginary component, except we omit the imaginary unit when specifying the number, since it is always with the imaginary unit.

27.3 Products of Complex Numbers with respect to their Components

Multiplying complex numbers by components, it is similar to a perfect square except the imaginary unit $i^2 = -1$ does not form a positive term when multiplying the even-number terms imaginary components. Hence let $z = x + iy$ and $w = u + iv$ then for zw

$$\begin{aligned} zw &= (x + iy)\,(u + iv) \\ &= xu + ixv + iuy - yv \end{aligned} \tag{27.28}$$

Meanwhile eq. (27.7) has explained either zw or wz may return of eq. (27.28). One useful complex number product involves with complex conjugates. It

[7] Index k to avoid confusion with the imaginary unit i.

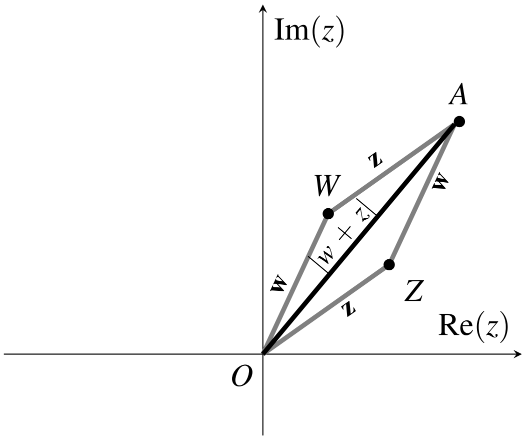

Figure 27.4: Addition of complex numbers z and w. Where z and w forms a vector in form of $\mathbf{z} \equiv \langle \mathrm{Re}(z), \mathrm{Im}(z) \rangle$ and $\mathbf{w} \equiv \langle \mathrm{Re}(w), \mathrm{Im}(w) \rangle$. When computing a vector addition, given that it is commutative (i.e. does not depend on the order of addition, and both returns the same answer), then a parallelogram is shown for visualisation.

does not limit to the product of conjugate pairs, but when multiply with complex conjugate of the other term can still be useful. For example, the latter is frequently used in rationalising fractions with complex numbers as its denominator. For two complex numbers $z = x + iy$ and $w = u + iv$ hence

$$\begin{aligned} zz* &= x^2 + y^2 = |z|^2 \\ zw* &= xu + vy + iyu - ixv \\ wz* &= xu + vy - iyu + ixv \\ ww* &= u^2 + v^2 = |w|^2 \end{aligned} \tag{27.29}$$

The first and last row of eq. (27.29) states that such product eliminates the imaginary component in the denominator.

27.4 Fractions involving Complex Numbers: Ratios of Complex Numbers

Complex numbers as a member of the numerical sets, could proceed divisions among more than one complex numbers. Fractions involving complex numbers are the ratio of two (2) complex numbers. For example, let $z = x + iy$ and $w = u + iv$ where $\{z, w\} \in \mathbb{C}$. The ratio of either cannot be converted to Cartesian form due to the denominator contains the imaginary component. To rationalise such, we may apply the products of conjugate pairs:

$$\frac{z}{w} = \frac{zw*}{ww*} \tag{27.30}$$

Given that the target system decides who could be in the denominator. For fairness, products involved with either $z*$ and $w*$ are listed in eq. (27.29) on page 169.

Exercise 27.4.1: Ratio of Complex Conjugate Pairs: z versus $z*$

Compute $\frac{z}{z*}$ where $z = x + iy$.

Solution:

$$\begin{aligned}\frac{z}{z*} &= \frac{x+iy}{x-iy} \\ &= \frac{x+iy}{x-iy}\frac{x+iy}{x+iy} \quad \text{(Multiply with an equivalent fraction to one (1))} \\ &= \frac{x^2+2ixy-y^2}{x^2+y^2}\end{aligned} \tag{27.31}$$

Exercise 27.4.2: Ratio of Complex Conjugate Pairs: $z*$ versus z

Compute $\frac{z*}{z}$ where $z = x + iy$.

Solution:

$$\begin{aligned}\frac{z*}{z} &= \frac{x-iy}{x+iy} \\ &= \frac{x-iy}{x+iy}\frac{x-iy}{x-iy} \quad \text{(Multiply with an equivalent fraction to one (1))} \\ &= \frac{x^2-2ixy-y^2}{x^2+y^2}\end{aligned} \tag{27.32}$$

27.5 Polar Representations of Complex Numbers: $re^{i\theta}$

Using multiple coordinates is due to **the target system is measured along the coordinate system**. Such as Cartesian coordinates are ideal for visualising vectors according to the linearly orthogonal coordinates, while polar coordinates are ideal for systems involves rotating elements.

For complex numbers, using polar coordinates simplifies when finding the roots of any complex numbers. Where in such case, let a complex number z be defined as

$$z = re^{i\theta} \tag{27.33}$$

where e^x is the **natural exponential function**.

A system polar coordinates contains the **magnitude** r which defined according to the complex number z as,

$$r = |z| \tag{27.34}$$

where therefore this is a scalar quantity. In writings, r is the **magnitude of the complex number**[8] z.

The polar angle of z is noted as the **argument of the complex number** z.

27.5.1 Definition (Argument of Complex Number z)
The argument of a complex number z is the angle between the vector z in the Argand diagram and the axis of real component. Noted as $\arg(z)$.

In this book, θ is the commonly pronumeral represents $\arg(z)$. When θ is occupied by other meanings, will note so separately.

27.5.1 Special Values for Arguments of Complex Numbers ± 1 and $\pm i$

Given that powers of the imaginary unit i computes either $\pm i$ or ± 1. By using specific powers of natural exponential function, we may mimic the same idea. That is,

$$\begin{aligned} i^0 &= e^0 &= 1 \\ i^1 &= e^{\frac{i\pi}{2}} &= i \\ i^2 &= e^{i\pi} &= -1 \\ i^3 &= e^{\frac{3i\pi}{2}} &= -i \\ i^4 &= e^{2i\pi} &= 1 \\ \vdots &\ \vdots & \vdots \\ i^n &= e^{\frac{i\pi n}{2}} \end{aligned} \tag{27.35}$$

[8] Specify z is a complex number is indicative to readers of what is z.

where n is not limited to any number, including complex numbers. The converse can be done as well,

$$\begin{aligned} \arg(1) &= 0 \\ \arg(i) &= \frac{\pi}{2} \\ \arg(-1) &= \pi \\ \arg(-i) &= -\frac{\pi}{2} \end{aligned} \tag{27.36}$$

where the last row of eq. (27.36) uses the range of radians of $(-\pi, \pi]$. Also, one may use eq. (27.37) to deduce eq. (27.36) for arguments outside the scoped domain.

27.5.2 A Complex Number has Many Arguments

The Argand Diagram defines the complex number into Polar Coordinates does not make an upper bound to the arguments of any complex numbers. However, given that the argument will return to the same complex number after rotating the vector in 360° (or 2π radians). We would wish to map complex numbers in polar form onto Cartesian form, so then the argument $\arg(z)$ must map onto the **principle argument of complex number** z $\text{Arg}(z)$. The arguments are related by

$$\text{Arg}(z) + 2\pi m = \arg(z) \quad : m \in \mathbb{C} \tag{27.37}$$

where using m corresponds to section 27.6.

Given that the projection of complex numbers with any full rotation does not change the value at all, then

$$\text{Arg}(z) + 2\pi m \equiv \text{Arg}(z) \quad : m \in \mathbb{C} \tag{27.38}$$

so that this persuades the reason for defining the principle argument. However, the use of principle argument will be explicit when finding roots of complex numbers or polynomial accepts complex field.

Also, when complex numbers rotate in 180°, hence π radians. This returns to the argument with the opposite parity.

$$\text{Arg}(z) + \pi m \equiv -\text{Arg}(z) \quad : m \in \mathbb{C} \tag{27.39}$$

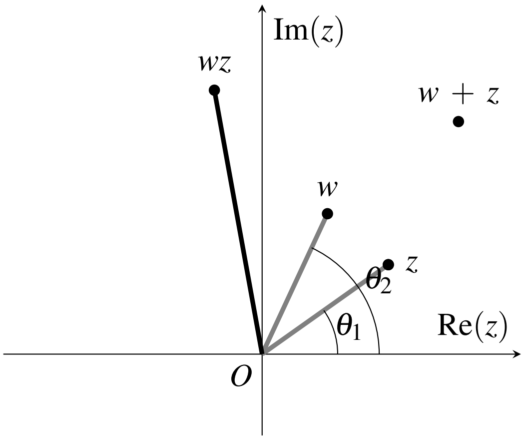

Figure 27.5: Multiplication of complex numbers z and w. The dot representing $z + w$ is a comparison to the product point zw.

27.5.3 Products of Complex Numbers in Polar Coordinates

Let $z_k = r_k\,\mathrm{e}^{i\theta_k}$ as one of the field vectors in the complex set. Where k is the index. Then from eq. (27.27)

$$\begin{aligned}\prod_k z_k &= \prod_k r_k\,\mathrm{e}^{i\theta_k}\\ &= \prod_{r_k} r_k \cdot \mathrm{e}^{i\sum_{\theta_k}\theta_k}\end{aligned} \tag{27.40}$$

where the last line of eq. (27.40) follows from the law of summations of exponents. eq. (27.40) is a different operation upon eq. (27.27): the earlier computes sum of polar angles while multiplying all of the complex (number) magnitudes. The latter is the sum of each individual components, as per eq. (27.27)states.

27.5.4 Multiplying Two Complex Numbers in Polar Coordinates

Usually we multiply only two complex numbers in one operation, then eq. (27.40) can be simplified to

$$\mathrm{e}^{i\theta_1}\,\mathrm{e}^{i\theta_2} = \mathrm{e}^{i(\theta_1+\theta_2)} \tag{27.41}$$

27.5.5 Multiplying Many Same of the Complex Numbers in Polar Coordinates

When multiplying complex numbers that are same in bulk amount. If one knows the number of terms, then

$$\left(\mathrm{e}^{i\theta}\right)^m = \mathrm{e}^{im\theta} \tag{27.42}$$

as per the exponential law of power of exponentials.

27.5.6 Ratios of Complex Numbers in Polar Form is an Inverse Operation to Products of Complex Numbers

When two complex numbers forms a ratio, polar form is still in its advantage. Given that two complex numbers multiply, their arguments add, vice versa. Hence the ratios of two complex numbers can be expressed in

$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|} e^{\theta_2 - \theta_1} \tag{27.43}$$

Note the order of subtracting the arguments in eq. (27.43).

27.5.7 Transformations of Complex Numbers in between Polar and Cartesian Coordinates

Figure 27.7 on the facing page references any complex numbers among the two coordinate systems:

- **Cartesian Coordinates** in terms of real and imaginary components.
- **Polar Coordinates** in terms of magnitudes and arguments.

For the mapping of $z(x, y) \mapsto z(r, \theta)$,

$$x = r\cos\theta \tag{27.44}$$

$$y = r\sin\theta \tag{27.45}$$

$$\tan\left(\frac{y}{x}\right) = \theta \tag{27.46}$$

where θ is not multivalued, since the complex number referenced is not.

For the mapping of $z(r, \theta) \mapsto z(x, y)$,

$$r^2 = x^2 + y^2 \tag{27.47}$$

$$\tan^{-1}\theta = \frac{y}{x} \tag{27.48}$$

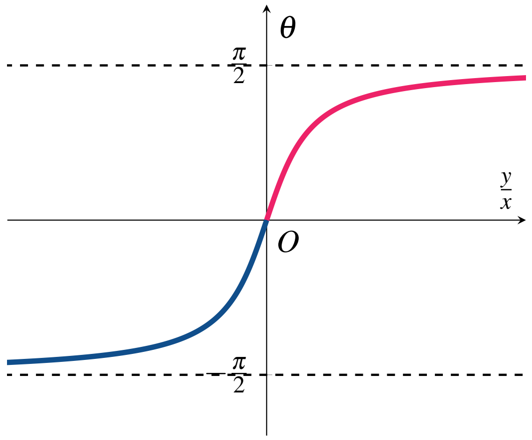

Figure 27.6: When mapping arguments onto the component values, note the parity of the argument. In the region of positive region of $\frac{y}{x}$ (in **Red**), each components must have the same parity. In the region of negative region of $\frac{y}{x}$ (in **Blue**), each components must differ in parities.

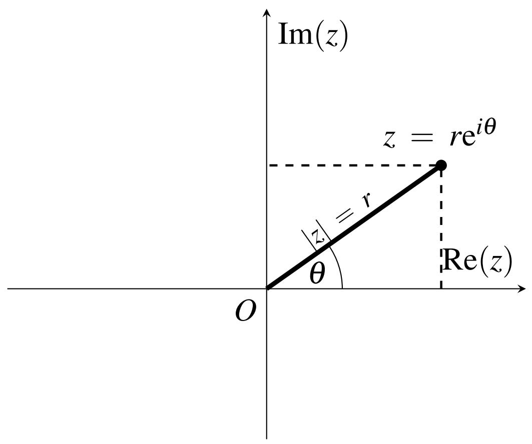

Figure 27.7: Polar representations of complex number $z : x + iy \mapsto re^{i\theta}$.

27.6 Roots of Complex Numbers

Algebra consists of studying the roots of functions and numbers. By means roots of numbers, for example the square root. Similar to the Fundamental Theorem of Algebra, the number of roots corresponds to the number of solutions. The polar form of any complex numbers, since it simplifies any product or division operations among all complex numbers, then for a complex number z with n roots

$$z^n = \sqrt[n]{|z|}\,\mathrm{e}^{\frac{im\theta}{n}} \quad : \{m, n\} \in \mathbb{Z} \tag{27.49}$$

Where θ is the pronumeral representing the principle argument $\mathrm{Arg}(z)$. Since m and n are unbounded, we iterate all m's until $m \geq n$.

27.6.1 de Moivre's Theorem

From the Euler's formula for complex numbers[9]

$$\mathrm{e}^{ix} = \cos x + i \sin x \quad x \in \mathbb{R} \tag{27.50}$$

Hence in the regime of integer powers of the exponential function, the exponential laws still holds but more powerful when also applying onto the right hand side of eq. (27.50).

$$\left(\mathrm{e}^{ix}\right)^n = \cos(nx) + i \sin(nx) \quad x \in \mathbb{R}, n \in \mathbb{N} \tag{27.51}$$

Equation (27.51) is the **de Moivre's theorem** or **de Moivre's formula**. For the polar form of complex number eq. (27.33), eq. (27.51) applies to the powers of complex numbers and transforms onto the form of sines and cosines of the argument of the complex number. So that for an arbitrary complex number z

$$z^n = \left(r\mathrm{e}^{i\theta}\right)^n = r^n \left[\cos(n\theta) + i \sin(n\theta)\right] \tag{27.52}$$

is valid.

27.7 Roots of Polynomials accepts Complex Set

27.7.1 Fundamental Theorem of Algebra

Before we further our discussions, fundamental theorem of algebra shapes what our solution could be. Here is the statement:

[9] Given that there are many formulas and equations are founded by Euler, this notation is just for avoiding confusions.

27.7.1 Theorem (Fundamental Theorem of Algebra)
Every non-zero, single-variable, degree n polynomial with complex coefficients has, counted with multiplicity, exactly n complex roots.

This theorem is means that for any particular polynomial, for example

$$x^2 + 1 = 0 \tag{27.53}$$

will have two (i.e. 2) solutions for x. When students start to learn complex numbers, they are introduced by solving different equations. While eq. (27.53) may prohibit them from substituting any (real) numbers they may think of. The theorem explicit tells that there must be two (i.e. 2) solutions. While complex numbers as the last salvage, means that when all numbers cannot help to solve polynomial equations like eq. (27.53), then seek for complex numbers.

Hence

$$\begin{aligned} x^2 + 1 &= 0 \\ x^2 &= -1 \\ x &= \pm\sqrt{-1} \\ x &= \pm i \end{aligned} \tag{27.54}$$

We have the two (i.e. 2) solutions for x then.

Bibliography

Wikipedia (2016). **Fundamental theorem of algebra — Wikipedia, The Free Encyclopedia**. en. [Online; accessed 13-January-2017]. URL: https://en.wikipedia.org/w/index.php?title=Fundamental_theorem_of_algebra&oldid=755976409.

Appendix 27.A Working Diary

27.A.1 13/01/2017

This chapter has been done before combining both notes. Theorem 27.7.1 is from Wikipedia (2016).

Start to do section 27.7.1 as a commitment to finish something started at the beginning of this project.

Appendix 27.B Selected Readings and their Reviews

Appendix 27.C Connections to Other Topics

Appendix 27.D Documentations

Chapter 28

Functions in the Complex Set

28.1 Logarithmic Functions in the Complex Set

When $z = re^{i\theta} \neq 0$ is a complex number. Where the argument of z is multivalued such that $\theta = \Theta + 2n\pi$ and n is an integer. The logarithmic of z is,

$$\ln z = \ln r + i(\Theta + 2n\pi) \tag{28.1}$$

28.1.1 Complex Exponents

The expression of z^w where either z and w accepts complex numbers, is possible. This is easier when using polar form of complex numbers.

28.2 Trigonometric Functions in the Complex Set

Functions such as $\sin x$ and $\cos x$, when accepting the complex set reminds the arguments in the polar form of complex numbers.

For $\sin z$ where $z \in \mathbb{C}$ is defined with

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \tag{28.2}$$

For $\cos z$ where $z \in \mathbb{C}$ is defined with

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \tag{28.3}$$

For $\tan z$ where $z = r e^{i\theta}$ is hence defined with

$$\tan z = \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} \tag{28.4}$$

Note that eqs. (28.2) to (28.4) contains the complex magnitude z.

28.2.1 Properties of Trigonometric Functions in the Complex Set

For example, the sine function is an odd function.

$$\sin(-z) = -\sin z \tag{28.5}$$

While the cosine function is an even function.

$$\cos(-z) = \cos z \tag{28.6}$$

Hence eqs. (28.2) and (28.3) works like the function when excludes the complex set.

28.3 Hyperbolic Functions in the Complex Set

Hyperbolic functions such as $\sinh x$ and $\cosh x$ are defined similarly to eqs. (28.2) and (28.3). For $\sinh z$ where $z = r e^{i\theta}$ is defined with

$$\sinh z = \frac{e^{z} - e^{-z}}{2} \tag{28.7}$$

Hence for $\cosh z$ where $z = r e^{i\theta}$ is defined with

$$\cosh z = \frac{e^{z} + e^{-z}}{2} \tag{28.8}$$

While for $\tanh z$ where $z = r e^{i\theta}$ is hence defined with

$$\tanh z = \frac{e^{z} - e^{-z}}{e^{z} + e^{-z}} \tag{28.9}$$

Note that eqs. (28.2) to (28.4) and (28.7) to (28.9) contains the complex magnitude z.

Appendix 28.A Working Diary

Appendix 28.B Selected Readings and their Reviews

Appendix 28.C Connections to Other Topics

Appendix 28.D Documentations